Rule-IV :- When $X = \sin ax$ or $\cos(ax)$ and $f(D)$ can be expressed as a function of D^2

i,e, $f(D) = F(D^2)$

then $\frac{1}{F(D^2)} \sin ax = \frac{1}{F(-a^2)} \sin ax$ provided $F(-a^2) \neq 0$

and $\frac{1}{F(D^2)} \cos(ax) = \frac{1}{F(-a^2)} \cos(ax)$ provided $F(-a^2) \neq 0$.

(*) If $F(-a^2)=0$ holds then

In this case we use $y+iz$ method.

Let $y = \frac{1}{F(D^2)} \cos ax$ and $z = \frac{1}{F(D^2)} \sin ax$.

So, $y + iz = \frac{1}{F(D^2)} (\cos ax + i \sin ax)$

$= \frac{1}{F(D^2)} e^{iax}$ (since $e^{i\theta} = \cos\theta + i\sin\theta$)

$= \frac{1}{f(D)} e^{iax}$

$= e^{iax} \frac{1}{f(D+ia)} \cdot 1$

$= P + iQ$ (say).

So seperating real and imaginary parts

$y = \frac{1}{f(D)} \cos ax = P(x)$ and

$z = \frac{1}{f(D)} \sin ax = Q(x)$.

Illustration

Calculate $\frac{1}{(D^2+1)} \sin x$.

Clearly if $F(D^2) = \frac{1}{(D^2+1)}$ then $F(-1)=0$, we have to use $y+iz$ method.

Let $y = \frac{1}{(D^2+1)} \cos x$ and $z = \frac{1}{(D^2+1)} \sin x$

now $y + iz = \frac{1}{(D^2+1)} (\cos x + i \sin x) = \frac{1}{(D^2+1)} e^{ix}$

$= e^{ix} \frac{1}{(D+i)^2+1} \cdot 1$

$$= e^{ix}\frac{1}{(D^2+2iD)}\cdot 1$$

$$= e^{ix}\cdot\frac{1}{2iD\left(1+\frac{D}{2i}\right)}\cdot 1$$

$$= \frac{e^{ix}}{2i}\cdot\frac{1}{D}\left(1+\frac{D}{2i}\right)^{-1}\cdot 1$$

$$= \frac{e^{ix}}{2i}\cdot\frac{1}{D}\left(1-\frac{D}{2i}+\cdots\right)\cdot 1$$

$$= \frac{e^{ix}}{2i}\cdot\frac{1}{D}(1).$$

$$= \frac{xe^{ix}}{2i} = \frac{x(\cos x+i\sin x)}{2i}$$

$$\Rightarrow y+iz = \frac{x\cos x}{2i}+\frac{x\sin x}{2}.$$

$$\Rightarrow y+iz = -\frac{x\cos x}{2}i+\frac{x\sin x}{2}$$

now equating real and imaginary part we get $y=\frac{x\sin x}{2}$ and $z=-\frac{x\cos x}{2}$.

$$\therefore \frac{1}{(D^2+1)}\cos x = \frac{x\sin x}{2} \text{ and}$$

$$\frac{1}{(D^2+1)}\sin x = -\frac{x\cos x}{2}.$$

Solve $\frac{1}{(D^2+4)}\sin 2x$.

Solⁿ: If $F(D^2)=(D^2+4)$ then $F(-4)=0$

so we have to use $y+iz$ method.

let $y = \frac{1}{(D^2+4)} \cos 2x$ and $z = \frac{1}{(D^2+4)} \sin 2x$

now $y+iz = \frac{1}{(D^2+4)} (\cos 2x + i \sin 2x)$

$$= \frac{1}{(D^2+4)} e^{2ix}$$

$$= e^{2ix} \cdot \frac{1}{(D+2i)^2+4} \cdot (1)$$

$$= e^{2ix} \frac{1}{D^2+4Di} \cdot (1)$$

$$= e^{2ix} \cdot \frac{1}{4Di\left(1+\frac{D}{4i}\right)} \cdot 1$$

$$= \frac{e^{2ix}}{4i} \cdot \frac{1}{D} \cdot \left(1+\frac{D}{4i}\right)^{-1} \cdot (1)$$

$$= \frac{e^{2ix}}{4i} \cdot \frac{1}{D} \cdot \left(1 - \frac{D}{4i} + \cdots\right) \cdot (1)$$

$$= \frac{e^{2ix}}{4i} \cdot \frac{1}{D} \cdot (1).$$

$$= \frac{e^{2ix}}{4i} x = \frac{x(\cos 2x + i \sin 2x)}{4i}$$

$$= \frac{x \cos 2x}{4i} + \frac{x \sin 2x}{4}$$

$y+iz$ $= -\frac{x \cos 2x}{4} i + \frac{x \sin 2x}{4}$.

now equating real and imaginary part we get

$y = \frac{1}{(D^2+4)} \cos 2x = \frac{x \sin 2x}{4}$ and

$z = \frac{1}{(D^2+4)} \sin 2x = -\frac{x \cos 2x}{4}$.

Solve

$$y'' + 4y = x\sin^2 x.$$

Solution:- The equation can be written as

$$(D^2+4)y = x\sin^2 x.$$

The auxiliary equation is $m^2+4=0 \Rightarrow m = \pm 2i$

so the complementary function is $(A\cos 2x + B\sin 2x)$

where A and B are arbitrary constants.

Now the P.I $= \dfrac{1}{(D^2+4)} x\sin^2 x$

$$= \frac{1}{2(D^2+4)}(x \times 2\sin^2 x)$$

$$= \frac{1}{2(D^2+4)} x(1-\cos 2x)$$

$$= \frac{1}{2(D^2+4)} \cdot x - \frac{1}{2(D^2+4)} x\cos 2x \quad \text{——} (1)$$

Now $\dfrac{1}{(D^2+4)} x$

$$= \frac{1}{4\left(1+\frac{D^2}{4}\right)} \cdot x$$

$$= \frac{1}{4}\left(1+\frac{D^2}{4}\right)^{-1} \cdot x = \frac{1}{4}\left(1-\frac{D^2}{4}+\cdots\right) \cdot x$$

$$= \frac{1}{4}(x) = \frac{x}{4}.$$

now for $\dfrac{1}{(D^2+4)}(x\cos 2x)$ we have to use

by $Y+iZ$ method.

Let $Y = \dfrac{1}{(D^2+4)} x\cos 2x$ and $Z = \dfrac{1}{(D^2+4)} x\sin 2x$.

now $y + iz = \frac{1}{(D^2+4)} x e^{2ix}$

$$= e^{2ix} \frac{1}{[(D+2i)^2+4]} \cdot x$$

$$= e^{2ix} \cdot \frac{1}{(D^2+4Di)} \cdot (x)$$

$$= e^{2ix} \frac{1}{4Di\left(1+\frac{D}{4i}\right)} \cdot (x)$$

$$= \frac{e^{2ix}}{4i} \cdot \frac{1}{D} \left(1+\frac{D}{4i}\right)^{-1} \cdot (x)$$

$$= \frac{e^{2ix}}{4i} \cdot \frac{1}{D} \left(1 - \frac{D}{4i} + \cdots\right) \cdot (x)$$

$$= \frac{e^{2ix}}{4i} \frac{1}{D} \left(x - \frac{1}{4i}\right)$$

$$= \frac{e^{2ix}}{4i} \left(\frac{x^2}{2} - \frac{x}{4i}\right)$$

$$= -\frac{(\cos 2x + i \sin 2x)\, i}{4} \left(\frac{x^2}{2} + \frac{xi}{4}\right)$$

$$= \left(-\frac{i\cos 2x}{4} + \frac{\sin 2x}{4}\right) \left(\frac{x^2}{2} + \frac{xi}{4}\right)$$

$$y + iz = \left(\frac{x\cos 2x}{16} + \frac{x^2 \sin 2x}{8}\right) + i\left(\frac{x \sin 2x}{16} - \frac{x^2 \cos 2x}{8}\right)$$

now $y = \frac{x\cos 2x}{16} + \frac{x^2 \sin 2x}{8}$ and

$$z = \frac{x \sin 2x}{16} - \frac{x^2 \cos 2x}{8}.$$

Now from (1) we get

$$P.I = \frac{1}{2}\left(\frac{x}{4}\right) - \frac{1}{2}\left(\frac{1}{16}\left(x\cos 2x + 2x^2 \sin 2x\right)\right).$$

$= -\frac{1}{32}x(\cos 2x + 2x\sin 2x - 4)$

$\therefore$ Required solution is

$y = A\cos 2x + B\sin 2x - \frac{1}{32}x(\cos 2x + 2x\sin 2x - 4).$

Solve

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = xe^x\sin x.$$

Solution:-

Here Auxiliary equation is $m^2 - 2m + 1 = 0$

$\Rightarrow m = 1, 1$

$\therefore$ C.F is $(A + Bx)e^x$ where $A, B =$ arbitrary const.

now $P.I. = \frac{1}{(D-1)^2} xe^x\sin x$

$= e^x \frac{1}{[(D+1)-1]^2} x\sin x$

$= e^x \frac{1}{D^2} x\sin x.$

$= e^x \cdot \frac{1}{D} \frac{1}{D}(x\sin x)$

$= e^x \frac{1}{D}(-x\cos x + \sin x)$

$= -e^x(x\sin x + 2\cos x)$

$$\boxed{\begin{array}{l}\text{note:- check by } Y + iZ \text{ method of } \frac{1}{D^2}x\sin x \\ \text{Here } Y = \frac{1}{D^2}x\cos x \; ; \; Z = \frac{1}{D^2}x\sin x\end{array}}$$

required solution $y = (A + Bx)e^x - e^x(x\sin x + 2\cos x).$

HW

① Solve $(D^2+4)(D^2+1)y = \cos 2x + \sin x$

Ans → $y = (c_1 \cos x + c_2 \sin x) + (c_3 \cos 2x + c_4 \sin 2x)$

$- \frac{x}{12}(\sin 2x + 2\cos x)$

P.I

② $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + \cancel{0} = e^x \sin x$. $\left[y = c_1 + c_2 e^{2x} - \frac{e^x \sin x}{2} \right]$

③ $\frac{d^2y}{dx^2} - y = x^2 \cos x$ $\left[y = c_1 e^x + c_2 e^{-x} + x\sin x + \frac{(1-x^2)\cos x}{2} \right]$

④ $\frac{d^2y}{dx^2} + 2y = x^2 e^{3x} + e^x \cos 2x$.

Ans → $P.I = \frac{e^{3x}}{121}\left(11x^2 - 12x + \frac{50}{11}\right) + \frac{e^x}{17}(4\sin 2x - \cos 2x)$